# ON PSEUDO-MERIDIANS OF THE TREFOIL KNOT GROUP (Representation spaces, twisted topological invariants and geometric structures of 3-manifolds)



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## 1. Introduction

Let $G(K)$ be the knot group of a knot $K$. We call a word $w\in G(K)$ a pseudo-meridian if $G(K)$ is normally generated by $w$, that is, $G(K)/\langle w\rangle$ is trivial where $\langle w\rangle$ is the normal closure of $w$ in $G(K)$. For example, a meridian of each knot group is a pseudo-meridian. Moreover, the image of a meridian under any automorphism of $G(K)$ is also a pseudo-meridian.

Silver-Whitten-Williams showed in [2] that the knot group $G(K)$ contains infinitely many non-equiavalent pseudo-meridians if $K$ is a non-trivial two bridge knot or a torus knot, or a hyperbolic knot with unknotting number one. Furthermore, they conjectured that every knot group has infinitely many non-equivalent pseudo-meridians.

In this short note, we will consider the trefoil knot $3_1$ and determine which word of $G(3_1)$ is a pseudo-meridian up to a certain word length.

## 2. Criterion

First, we fix the following presentation of the knot group of the trefoil:

$$G(3_1)=\langle x,y\,|\,xyx=yxy\rangle.$$

The generators $x$ and $y$ are meridians. Under this presentation, we investigate which word of $G(3_1)$ is a pseudo-meridian.

If $x$ or $y$ can be written as a product of conjugates of a word $w$ and the inverse $\bar{w}$ in $G(3_1)$, then $x$ and $y$ belong to the normal closure $\langle w\rangle$. Therefore $w$ is a pseudo-meridian. For example, $xx\bar{y}$ is a pseudo-meridian, since

$$x(xx\bar{y})\bar{x}\cdot\bar{y}(xx\bar{y})y\cdot\bar{x}\overline{(xx\bar{y})}x=xxx\bar{y}\bar{x}\bar{y}xy\bar{x}=xxx\bar{x}\bar{y}\bar{x}xy\bar{x}=x.$$

Here $\bar{z}$ is the inverse of $z$.

On the other hand, if the exponent sum of a word $w$ is neither 1 nor $-1$, then $x$ and $y$ cannot be written as a product of conjugates of $w$ and $\bar{w}$ in $G(3_1)$. Hence $w$ is not a pseudo-meridian. In addition, the follewing is a useful criterion to show that a word is not a pseudo-meridian.

**Lemma 2.1.** *Let $w$ be a word of $G(3_1)$. If there exists a non-trivial representation $\rho:G(3_1)\rightarrow SL(2;\mathbb{Z}/p\mathbb{Z})$ such that $\rho(w)$ is the identity matrix, then $w$ is not a pseudo-meridian.*

*Proof.* By the assumption that $\rho(w)$ is the identity matrix, $\rho$ factors through $G(3_1)/\langle w\rangle$. Namely, $\rho$ induces a representation

$$\bar{\rho}:G(3_1)/\langle w\rangle\longrightarrow SL(2;\mathbb{Z}/p\mathbb{Z}).$$

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| | pseudo-meridians |
|---|---|
| 1 | $x$, $y$ |
| 3 | $xx\bar{y}, \bar{x}yy$ |
| 5 | $xxx\bar{y}\bar{y}, xxy\bar{x}\bar{y}, xx\bar{y}x\bar{y}, xx\bar{y}\bar{x}y, xyx\bar{y}\bar{y}, xy\bar{x}\bar{x}y, xyy\bar{x}\bar{y}, x\bar{y}\bar{x}yy, \bar{x}\bar{x}yyy, \bar{x}y\bar{x}yy$ |
| 7 | $xxxx\bar{y}\bar{x}\bar{y}, xxxx\bar{y}\bar{y}\bar{y}, xxxy\bar{x}\bar{x}\bar{y}, xxxy\bar{x}\bar{y}\bar{y}, xxx\bar{y}x\bar{y}\bar{y}, xxx\bar{y}\bar{x}\bar{x}y, xxx\bar{y}\bar{y}x\bar{y}, xxx\bar{y}\bar{y}\bar{x}y,$<br>$xxyx\bar{y}\bar{x}\bar{y}, xxyx\bar{y}\bar{y}\bar{y}, xxy\bar{x}y\bar{x}\bar{y}, xxy\bar{x}\bar{y}x\bar{y}, xxy\bar{x}\bar{y}\bar{x}y, xxyy\bar{x}\bar{x}\bar{y}, xxyy\bar{x}\bar{y}\bar{y}, xx\bar{y}xx\bar{y}\bar{y},$<br>$xx\bar{y}xy\bar{x}\bar{y}, xx\bar{y}x\bar{y}x\bar{y}, xx\bar{y}x\bar{y}\bar{x}y, xx\bar{y}\bar{x}\bar{x}yy, xx\bar{y}\bar{x}yx\bar{y}, xx\bar{y}\bar{x}y\bar{x}y, xx\bar{y}\bar{x}\bar{y}xy, xx\bar{y}\bar{y}\bar{x}yy,$<br>$xx\bar{y}\bar{y}\bar{y}xy, xyxy\bar{x}\bar{x}\bar{y}, xyxy\bar{x}\bar{y}\bar{y}, xyx\bar{y}x\bar{y}\bar{y}, xyx\bar{y}\bar{x}\bar{x}y, xyx\bar{y}\bar{y}x\bar{y}, xyx\bar{y}\bar{y}\bar{x}y, xy\bar{x}\bar{x}\bar{x}yy,$<br>$xy\bar{x}\bar{x}y\bar{x}y, xy\bar{x}y\bar{x}\bar{x}y, xy\bar{x}yy\bar{x}\bar{y}, xy\bar{x}\bar{y}\bar{x}yy, xyyx\bar{y}\bar{x}\bar{y}, xyy\bar{x}\bar{x}\bar{x}y, xyy\bar{x}y\bar{x}\bar{y}, xyy\bar{x}\bar{y}x\bar{y},$<br>$xyy\bar{x}\bar{y}\bar{x}y, xyyy\bar{x}\bar{x}\bar{y}, xyyy\bar{x}\bar{y}\bar{y}, x\bar{y}x\bar{y}\bar{x}yy, x\bar{y}\bar{x}\bar{x}yyy, x\bar{y}\bar{x}y\bar{x}yy, x\bar{y}\bar{x}yy\bar{x}y, x\bar{y}\bar{y}\bar{x}yyy,$<br>$\bar{x}\bar{x}\bar{x}yyyy, \bar{x}\bar{x}y\bar{x}yyy, \bar{x}\bar{x}yy\bar{x}yy, \bar{x}\bar{x}yyy\bar{x}y, \bar{x}y\bar{x}y\bar{x}yy, \bar{x}yyyy\bar{x}\bar{y}$ |

| | non-pseudo-meridians |
|---|---|
| 7 | $xxy\bar{x}\bar{x}\bar{x}y, xy\bar{x}\bar{x}yx\bar{y}, xy\bar{x}yx\bar{y}\bar{y}, xyyx\bar{y}\bar{y}\bar{y}$ |

TABLE 1. pseudo-meridians and non-pseudo-meridians

Since $\rho$ is a non-trivial representation, $\rho(x)$ and $\rho(y)$ are not the identity matrix and then $\bar{\rho}(x), \bar{\rho}(y)$ are not the identity matrix too. Therefore $\bar{\rho}$ is also a non-trivial representation and $G(3_1)/\langle w\rangle$ is not trivial. This completes the proof. $\square$

For example, there exists a non-trivial representation

$$\rho : G(3_1) \longrightarrow SL(2; \mathbb{Z}/5\mathbb{Z})$$

defined by

$$\rho(x) = \begin{pmatrix} 0 & 1 \\ 4 & 3 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} 0 & 4 \\ 1 & 3 \end{pmatrix}.$$

It is easy to see that $\rho(xxy\bar{x}\bar{x}\bar{x}y)$ is the identity matrix. Then $xxy\bar{x}\bar{x}\bar{x}y$ is not a pseudo-meridian, though the exponent sum is 1.

## 3. MAIN RESULT

By using the method shown in Section 2, we obtain Table 1 which shows pseudo-meridians and non-pseudo-meridians up to word length 7. The first column on Table 1 is word length.

All words whose exponential sum are not $\pm 1$ are not pseudo-meridians and then we enumerate only the words whose exponential sum are $\pm 1$. If a word is a pseudo-meridian, then the cyclic words and the inverses are also pseudo-meridians. For instance, $xx\bar{y}$ is a pseudo-meridian and then $x\bar{y}x, \bar{y}xx, y\bar{x}\bar{x}, \bar{x}\bar{x}y, \bar{x}y\bar{x}$ are so. The converse statement is also true. Therefore one of them is listed in Table 1. Besides, $xxyx\bar{y}\bar{x}\bar{y}$ is same as $x$ for example. However, both of them are listed.

## 4. PROBLEMS

In Section 3, we determined which words of $G(3_1)$ up to the word length 7. Next, we want to consider the following.

**Problem 4.1.** *Determine which word of $G(3_1)$ is a pseudo-meridian under the fixed presentation.*

In this note, we deal only with the trefoil. However, we would like to consider all knot groups.

**Problem 4.2.** *Characterize the words of pseudo-meridians for given knot groups. In other words, find a useful criterion to determine whether a word is a pseudo-meridian or not.*

## References

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